

The Energy of an Electron Moving along a Rectilinear Trajectory in the Vacuum

Gaobiao Xiao

Full Professor, Shanghai Jiao Tong University, China

gaobiaoxiao@sjtu.edu.cn

Abstract— The self-force and the self-energy (Coulomb-velocity energy) of an electron moving along a rectilinear trajectory in the vacuum is analyzed numerically. It is illustrated for the first time that when the velocity of the electron approaches the light velocity in the vacuum, the Coulomb-velocity energy approaches infinitely large, so is the self-force. Consequently, electrons cannot be accelerated to move faster than the light in the vacuum using electric accelerators because infinite large external force may be required to make the electron cross the electromagnetic barrier of the light velocity in the vacuum. Based on the observation, the Bertozzi experiment is re-interpreted, which shows that the velocity limit is due to the intrinsic behavior of the electron that can be clearly explained with the classical Maxwell's theory. It is obviously not so definite that this behavior of the electron is due to the effect of special relativity as claimed in text books. Therefore, the outcome of the Bertozzi experiment may be not an unquestionable experimental support to the Einstein's theory of special relativity. It is natural to consider that neutral particles may move faster than the light velocity in the vacuum because they do not face the big electromagnetic self-force when they cross the electromagnetic barrier. Furthermore, a reasonable hypothesis can be made that superluminal electrons may be generated by the collision of high energy particles in a collider or in the universe.

Index Terms— Electromagnetic self-force, self-energy, Coulomb-velocity energy, electromagnetic barrier, moving electron, Bertozzi experiment.

I. THE SELF-FORCE AND THE SELF-ENERGY OF AN ELECTRON

As pointed out in [1], the classical theory of charged particles was primary a theory of the electron. The most important theory for particle dynamics is perhaps developed by H. A. Lorentz [2][3], the basis of which is that the charged particles do not interact with each other directly but via the electromagnetic fields.

An electron can be modeled with a charge density $\rho(\mathbf{r}_1)$ within a sphere V_e with radius r_e . It may be modeled with a uniformly distributed charge ball or a charge shell. Its total charge e can be calculated by

$$e = \int_{V_e} \rho(\mathbf{r}_1) d\mathbf{r}_1 \quad (1)$$

The electron generates electromagnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ in the space. In the vacuum, the electromagnetic fields can be evaluated with the corresponding potentials:

$$\begin{cases} \mathbf{E}(\mathbf{r}, t) = -\nabla\phi(\mathbf{r}, t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t) \\ \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \end{cases} \quad (2)$$

For a particle instantaneously at rest, its velocity \mathbf{v} is zero, the scalar potential $\phi(\mathbf{r}, t)$ and the vector potential $\mathbf{A}(\mathbf{r}, t)$ are expressed by

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{V_e} \frac{\rho(\mathbf{r}_1, t - R/c_0)}{R} d\mathbf{r}_1 \quad (3)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V_e} \frac{\mathbf{J}(\mathbf{r}_1, t - R/c_0)}{R} d\mathbf{r}_1 = \frac{1}{4\pi\epsilon_0 c_0^2} \int_{V_e} \frac{\mathbf{J}(\mathbf{r}_1, t - R/c_0)}{R} d\mathbf{r}_1 \quad (4)$$

where $R = |\mathbf{r} - \mathbf{r}_1| = c_0 t$, c_0 is the light velocity in the vacuum, μ_0 and ϵ_0 are respectively the permeability and the permittivity in the vacuum. Throughout this paper, we generally use (\mathbf{r}, t) for the field points or observation points and (\mathbf{r}_1, t_1) for the sources if not declared otherwise.

For a moving electron, the evaluation of the self-force is more complicated. The electron can be decomposed into point charges of $\rho d\mathbf{r}_1$. The potentials of the electron can be obtained by integrating the Liénard-Wiechert potentials of $\rho d\mathbf{r}_1$ [4]-[7],

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{V_e} \frac{\rho(\mathbf{r}_1, t_1)}{R[1 - \hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}}(t_1)]} d\mathbf{r}_1 \quad (5)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c_0} \int_{V_e} \frac{\rho(\mathbf{r}_1, t_1) \boldsymbol{\beta}(t_1)}{R[1 - \hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}}(t_1)]} d\mathbf{r}_1 \quad (6)$$

where $R = |\mathbf{r}(t) - \mathbf{r}_1(t_1)| = c_0(t - t_1)$, $\hat{\mathbf{n}} = \mathbf{R}/R$, and $\mathbf{R} = \mathbf{r}(t) - \mathbf{r}_1(t_1)$. $\boldsymbol{\beta} = \mathbf{v}/c_0$ denotes the normalized velocity. The electric field and the magnetic field of the electron are derived using exactly the same method as in deriving the fields of a moving point charge [7](21-47, Panofsky)

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{V_e} \left[\frac{(1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta})(\hat{\mathbf{n}} - \boldsymbol{\beta})}{R^2(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} + \frac{\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{c_0 R(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} \right] \rho(\mathbf{r}_1, t_1) d\mathbf{r}_1 \quad (7)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c_0} \int_{V_e} \left\{ \hat{\mathbf{n}} \times \left[\frac{(1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta})(\hat{\mathbf{n}} - \boldsymbol{\beta})}{R^2(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} + \frac{\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{c_0 R(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} \right] \right\} \rho(\mathbf{r}_1, t_1) d\mathbf{r}_1 \quad (8)$$

Note that $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$, $\hat{\mathbf{n}}$ and R in the right-hand side of the expressions are evaluated at the retarded time t_1 . We use “.” on top of a variable to denote its derivation with respect to the time. Hence, the acceleration can be expressed by $\mathbf{a} = c_0 \dot{\boldsymbol{\beta}}$. For the field at (\mathbf{r}, t) , the retarded time t_1 for the source at point \mathbf{r}_1 can be determined with [8]

$$R = c_0(t - t_1) = |\mathbf{r}(t) - \mathbf{x}(t_1)| \quad (9)$$

In particular, the solution of t_1 must obey the causality principle, that is,

$$t_1 \text{ must be real-valued, and } t_1 \leq t \quad (10)$$

All other solutions for t_1 of (9) that do not meet the requirement of the causality principle (10) are discarded.

As pointed out previously, an electron may be modeled with rigid structures, such as a ball with a uniform volume charge density or a shell with a uniform surface charge density. When we have selected a model for the electron, we may need to fulfil the following tasks to illustrate the behavior of the electron in the vacuum:

- (1) For a known trajectory, to evaluate the electromagnetic fields of the electron at any point (\mathbf{r}, t) ;
- (2) When the external force acting on the electron is known, to determine the trajectory of the electron;
- (3) When the conditions at $t \leq t_0$ are known, to determine the trajectory of the electron for $t \geq t_0$ without external force.

We will show that by making use of the causality principle, the task-1 can be evaluated rigorously with proper numerically methods. The task-2 and task-3 are usually combined together. They are basically nonlinear dynamic systems. We are to develop the dynamic equations to describe their behavior in the following sections.

A. The self-force of the electron

The electromagnetic self-force that the fields exert on the electron itself is calculated by

$$\mathbf{F}_{self}(t) = \int_{V_e} \rho(\mathbf{r}, t) [\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)] d\mathbf{r} \quad (11)$$

The integrand in the righthand side of (11) is the Lorentz force density,

$$\mathbf{f} = \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (12)$$

Historically, the self-force of an electron is first approximately evaluated by Lorentz [2][4]. Since the electron is very small, $R/c \ll 1$ holds. The retarded sources can be expanded in the vicinity of the current time t as,

$$\rho\left(\mathbf{r}_1, t - \frac{R}{c_0}\right) \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{R}{c_0}\right)^n \frac{\partial^n}{\partial t^n} \rho(\mathbf{r}_1, t) \quad (13)$$

$$\mathbf{J}\left(\mathbf{r}_1, t - \frac{R}{c_0}\right) \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{R}{c_0}\right)^n \frac{\partial^n}{\partial t^n} \mathbf{J}(\mathbf{r}_1, t) \quad (14)$$

Following the detailed process given in [4], the self-force is found to be

$$\begin{aligned} \mathbf{F}_{self} &= -\frac{1}{6\pi\epsilon_0 c_0^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n \mathbf{a}}{c_0^n dt^n} \int_{V_e} \int_{V_e} \rho(\mathbf{r}) \rho(\mathbf{r}_1) R^{n-1} d\mathbf{r}_1 d\mathbf{r} \\ &\approx -\frac{1}{3\pi\epsilon_0 c_0^2} W_{self} \mathbf{a} + \frac{e^2}{6\pi\epsilon_0 c_0^3} \dot{\mathbf{a}} - \frac{e^2}{6\pi\epsilon_0 c_0^2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{d^n \mathbf{a}}{c_0^n dt^n} O(R^{n-1}) \end{aligned} \quad (15)$$

In the rest frame, the self-energy is simply expressed by

$$W_{self} = \frac{1}{8\pi\epsilon_0} \int_{V_e} \int_{V_e} \frac{\rho(\mathbf{r})\rho(\mathbf{r}_1)}{R} d\mathbf{r}_1 d\mathbf{r} \quad (16)$$

For very small r_e , the terms containing $O(R^{n-1}) \approx 0$ for $n \geq 2$, so the self-force is approximately expressed by:

$$\mathbf{F}_{self} \approx -\frac{4}{3c_0^2} W_{self} \mathbf{a} + \frac{e^2}{6\pi\epsilon_0 c_0^3} \dot{\mathbf{a}} \quad (17)$$

In particular, since $W_{self} = e^2/(8\pi\epsilon_0 r_e)$ for a uniform charge shell, the damping force by the electromagnetic radiation is explicitly introduced by Lorentz as

$$\mathbf{F}_{rad} = -\frac{e^2}{6\pi\epsilon_0 c_0^3} \dot{\mathbf{a}} \quad (18)$$

Equation (18) is the well-known Abraham-Lorentz radiation reaction formula [4]. It is related to the famous Larmor formula [9] for the electromagnetic radiation power of an accelerated charged particle:

$$\int_{t_1}^{t_2} \mathbf{F}_{rad} \cdot \mathbf{v} dt = \int_{t_1}^{t_2} P(t) dt = \int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_0 c_0^3} \mathbf{a} \cdot \mathbf{a} dt \quad (19)$$

When an external force \mathbf{F}_{ext} acts on the electron, it has a twofold contribution. One is to push the electron forward based on the Newton's law of motion. The other is to counter the radiation damping force. The equation of motion becomes

$$\mathbf{F}_{ext} = m_e \mathbf{a} - \mathbf{F}_{rad} = m_e \mathbf{a} - \frac{e^2}{6\pi\epsilon_0 c_0^3} \dot{\mathbf{a}} \quad (20)$$

which is the famous Abraham-Lorentz equation of motion. m_e is the mass of the electron. If the external force is zero, equation (20) has two solutions [4],

$$\begin{cases} \mathbf{a} = 0 \\ \mathbf{a} = e^{\alpha t} \end{cases} \quad (21)$$

where $\alpha = e^2/(6m_e\pi\epsilon_0 c_0^3)$. The first solution is stable and reasonable, the second solution is the runaway solution that has bothered many researchers in the early 1900s. Note that the radiation damping force (18) is an approximate expression for the self-force that is valid only for small velocity. When the electron keeps being accelerated, its velocity becomes large and the motion equation (20) is not applicable for the electron anymore. Techniques like pre-acceleration cannot overcome this difficulty satisfactorily from the root.

In this paper, we will adopt the uniform charged ball model for the electron and accurately calculate and the self-force and the self-energy by numerically evaluating the double-fold integrations. In particular, we have taken the Coulomb-velocity energy [10][11] as the self-energy. We will show later that the runaway solutions never happen because the radiation damping force increases with the velocity and approaches infinity when the velocity approaches c_0 .

Assume that the electron has uniform charge density of ρ_0 in the sphere V_e with radius r_e . Inserting (7) and (8) into (11), the self-force of the electron can be evaluated with

$$\begin{aligned} \mathbf{F}_{self} &= \frac{\rho_0^2}{4\pi\epsilon_0} \int_{V_e} \left\{ \mathbf{E}(\mathbf{r}, t) + \boldsymbol{\beta}(t) \times [\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r}, t)] \right\} d\mathbf{r} \\ &= \frac{\rho_0^2}{4\pi\epsilon_0} \int_{V_e} \int_{V_e} \left\{ \left[\frac{(1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta})(\hat{\mathbf{n}} - \boldsymbol{\beta})}{R^2 (1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} + \frac{\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})}{c_0 R (1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} \right]_{t_1} \right. \\ &\quad \left. - \boldsymbol{\beta}(t) \times \left[\frac{(1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta})(\hat{\mathbf{n}} \times \boldsymbol{\beta})}{R^2 (1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} + \frac{(\hat{\mathbf{n}} \times \boldsymbol{\beta})(\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}}) + (\hat{\mathbf{n}} \times \dot{\boldsymbol{\beta}})(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})}{c_0 R (1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} \right]_{t_1} \right\} d\mathbf{r}_1 d\mathbf{r} \end{aligned} \quad (22)$$

Take care that $\boldsymbol{\beta}(t)$ is evaluated at the time t while the quantities inside the square brackets are evaluated at t_1 .

We use equation (22) for the numerical evaluation. It is worth to emphasize that

- 1) When we have selected the model for the electron, the self-force is only dependent on the velocity $\boldsymbol{\beta}$ and the acceleration $\mathbf{a} = c_0 \dot{\boldsymbol{\beta}}$. It does not explicitly depends on $\dot{\mathbf{a}}$ as implied by (18). We may rewrite $\mathbf{F}_{self} = \mathbf{F}_{self}(\mathbf{v}, \mathbf{a})$ to highlight that the self-force is basically a nonlinear function with respect to both \mathbf{v} and \mathbf{a} .
- 2) The self-force is evaluated always in the laboratory frame. (22) is theoretically valid for all $\boldsymbol{\beta}$, including $\beta \geq 1$.

- 3) It can be proved that the self-force is exactly zero if the acceleration is zero. This property can be used as a criterion for evaluating the accuracy of the numerical algorithm.

B. The self-energy of the electron

Based on the Maxwell's equations, the energy of a pulse radiator in the vacuum can be divided into three parts [10][11]:

$$W_{tot}(t) = W_{\rho J}(t) + W_S(t) + W_{rad}(t) \quad (23)$$

The first part $W_{\rho J}(t)$ is the Coulomb-velocity energy and $W_{\rho J}(t) = W_\rho(t) + W_J(t)$, in which the Coulomb energy $W_\rho(t)$ and the velocity energy $W_J(t)$ are respectively defined as,

$$W_\rho(t) = \int_{V_c} \frac{1}{2} \rho(\mathbf{r}, t) \phi(\mathbf{r}, t) d\mathbf{r} \quad (24)$$

$$W_J(t) = \int_{V_c} \frac{1}{2} \mathbf{J}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t) d\mathbf{r} \quad (25)$$

It has been verified in [10][11] that the Coulomb-velocity energy is strictly attached to its source. When the source appears, the Coulomb-velocity energy appears; when the source disappears, the Coulomb-velocity energy disappears simultaneously with its source.

The second part $W_S(t)$ is the macroscopic Schott energy, which is defined by

$$W_S(t) = \int_{V_\infty} \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{D} \cdot \mathbf{A}) d\mathbf{r} \quad (26)$$

We have proved in [10][11] that after the source has disappeared, the Schott energy $W_S(t)$ does not disappear immediately. It becomes zero after a short while. The Schott energy represents a kind of spatial oscillation of the radiation energy that is caused by the superposition of the radiation fields in different directions.

The third part $W_{rad}(t)$ is the radiation energy defined by

$$W_{rad}(t) = - \int_{V_\infty} \left(\mathbf{D} \cdot \frac{\partial \mathbf{A}}{\partial t} \right) d\mathbf{r} \quad (27)$$

After being emitted, the radiation energy will leave its sources and keep propagating in the space until it encounters other sources and interacts with them. It continues to exist after the source has disappeared.

Obviously, the radiation energy $W_{rad}(t)$ and the macroscopic Schott energy $W_S(t)$ are not attached to their source. However, the macroscopic Schott energy does not behave like the radiation energy that keeps propagating to regions far away from the source region. It may propagate inwardly and interact with other part of sources in the source region repeatedly. Only after the sources have disappeared, $W_S(t)$ stops the energy exchanging and propagates to far regions like the radiation energy.

Based on their temporal evolution behavior, we treat the Coulomb-velocity energy of the electron as its self-energy. A more convenient strategy is to simply divide the energy of the electron into two parts, the Coulomb-velocity energy $W_{\rho J}(t)$ and an emitted energy $W_{emit}(t) = W_S(t) + W_{rad}(t)$, namely,

$$W_{tot}(t) = W_{\rho J}(t) + W_{emit}(t) \quad (28)$$

The Coulomb-velocity energy is attached to the electron, while the emitted energy $W_{emit}(t)$ is not. It leaves the electron after being emitted. It is important to note that, for moving electrons, the potentials have to be calculated with (5) and (6).

The energy separation equation (23) and (28) are directly derived from the Maxwell equations with no approximation. Although derived using pulse radiators, they are valid for analyzing the electromagnetic fields of a moving electron over any time period. If the electron keeps accelerating, it keeps radiating, and the total radiative electromagnetic energy keeps increasing and becomes unbounded. However, the total Coulomb-velocity energy will be determined with (24) and (25). We will show that it is bounded if $\beta \neq 1$.

C. The combined mechanical and electromagnetic system of the electron

It is widely considered that the electric charge e of the electron is a constant, while the mass of the electron may vary with its moving velocity, so we use the rest mass m_e . In this paper, the electric charge e and the mass m_e are two intrinsic quantities that are constant in all situations, so is the charge-mass ratio e/m_e .

As has pointed out previously, when an external force \mathbf{F}_{ext} exerts on an electron, it always divides into two parts and plays a double-fold role: one part (\mathbf{F}_{mech}) is to push the electron moving and the other part (\mathbf{F}_{em}) is to counter the electromagnetic self-force. The mechanical behavior of the electron obeys the Newtonian mechanics. The motion equation is,

$$\mathbf{F}_{mech} = m\mathbf{a} \quad (29)$$

The electromagnetic behavior of the electron obeys the Maxwell's equation. The self-force (Lorentz force) bridges the mechanic system and the electromagnetic system of the electron with

$$\mathbf{F}_{em} = -\mathbf{F}_{self} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \triangleq \mathbf{F}_{em}(\mathbf{v}, \mathbf{a}) \quad (30)$$

If $\mathbf{F}_{em} \cdot \mathbf{v} > 0$, then the force \mathbf{F}_{em} does work to convert the mechanical energy into the electromagnetic energy, namely, the Coulomb-velocity energy, the radiation energy, and the Schott energy. If $\mathbf{F}_{em} \cdot \mathbf{v} < 0$, then the force \mathbf{F}_{em} does work to convert the electromagnetic energy into mechanical energy.

The motion equation of the electron should be modified as

$$\mathbf{F}_{ext} = m\mathbf{a} + \mathbf{F}_{em}(\mathbf{v}, \mathbf{a}) \quad (31)$$

For the sake of simplicity, we only consider translation of the electron and have ignored the possible rotation and the role of the torques.

Note that the charge-mass ratio of the electron is a constant. If the velocity \mathbf{v} and the external force \mathbf{F}_{ext} are given at time t , the acceleration \mathbf{a} at time t can be uniquely determined with (31), so the trajectory of the electron can be determined step by step in the time domain. It can be expected that no runaway solutions exist because of the behaviors of \mathbf{F}_{em} that will be illustrated in the next section.

As a short summary, we remark that:

- 1) Equation (31) may be not a pleasant form because it does not look like the Newtonian motion equation. However, we have to make it clear that it is (31) that correctly describes the original dynamic property of the electron. At the beginning of developing the electromagnetic theory for moving objects, many difficulties were aroused simply because many researchers were too indulging in handling the electromagnetic problems in analogous with the Newtonian mechanics theory, and might have perhaps overlooked the importance of the essential fact that the electromagnetic issues are subject to the Maxwell's theory instead of the Newtonian mechanics. We will show that the nature of the electromagnetic phenomena may be illustrated more accurately and intuitively by rechecking the original description of the dynamic system of the electron using the Maxwell's theory.
- 2) The mechanical system and the electromagnetic system of the electron are connected only through the motion equation (31), with the same velocity \mathbf{v} , the same acceleration \mathbf{a} and a constant charge-mass ratio e/m_e . When the forces \mathbf{F}_{em} and \mathbf{F}_{mech} are determined, the other quantities of the two systems can be determined independently, including the energies, the momentums, and the fields, as shown in Fig. 1. The energy conservation and momentum conservation are always observed separately in the two systems. It is not necessary to describe the two systems as a whole in a similar formulation. For example, when \mathbf{v} and \mathbf{a} are determined, the source property of the electromagnetic system is determined as e is known. We can calculate the electromagnetic fields and energies from them simply according to the Maxwell's equations. Meantime, based on \mathbf{v} , \mathbf{a} , and m_e , we can calculate the kinetic energy with the classic mechanics laws. If necessary, we can get the total energy of the electron by combing the mechanical energy and the electromagnetic energy together.
- 3) The strategy of handling the two systems separately also shows that it is not necessary to define an electromagnetic mass for the electron, hence, we need not pay attention to the troubles like the 4/3 factor issue that has bothered so many researchers [1].
- 4) Most importantly, the conventional formulations, like the Abraham-Lorentz motion equation, are all based on some kind of approximate expressions that are only valid for low velocity situation. No papers can be found in literatures that can provide correct and wide-span descriptions for the behavior of the electron when the moving velocity approaches the light velocity in the vacuum. As we will show later, the conventional theory failed to catch the critical property that the self-force and the self-energy of the electron may approach infinity when the velocity approaches the light velocity in the vacuum, which we consider as one of the key reasons why the researchers in the early 1900s had met great difficulties in developing the electromagnetic theory for moving media. The electron with different velocity may have different self-energy. The concept of the transverse mass and the longitudinal mass only make sense if we try to handle the electromagnetic problems in analogous to the mechanics problems.

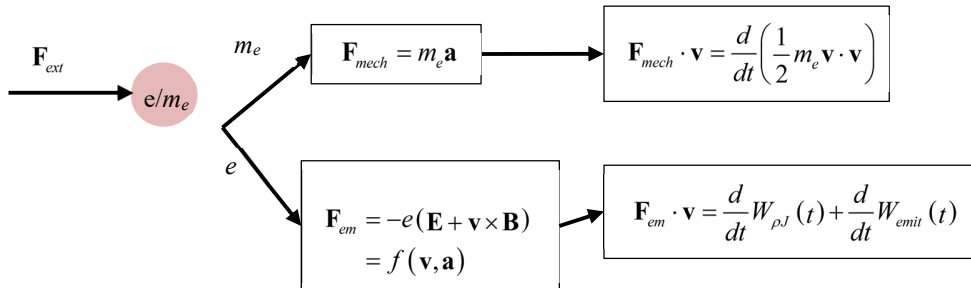


Fig. 1 Diagram of the combined mechanical and electromagnetic system of the electron.

II. NUMERICAL RESULTS OF THE SELF-FORCE AND THE SELF-ENERGY OF AN ELECTRON

We have to make use of the numerical techniques developed by the computational community to accurately evaluate the self-force and the self-energy. We here only consider the case that the electron moves along a rectilinear trajectory in $+z$ direction without any rotations.

Assume that, for $\beta < 1$, the electron is modeled with a rigid charge shell with radius r_e . Its center moves along a rectilinear trajectory $\mathbf{s}(t)$ in the vacuum and there is no deformation in the motion. We denote $\mathbf{v}(t) = c_0 \dot{\boldsymbol{\beta}}(t) = \dot{\mathbf{s}}(t)$, $\mathbf{a}(t) = c_0 \ddot{\boldsymbol{\beta}}(t)$. The coordinate for the field point and the source point can be expressed by

$$\begin{cases} \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{s}(t) \\ \mathbf{r}_1(t_1) = \mathbf{r}_{10} + \mathbf{s}(t_1) \end{cases} \quad (32)$$

For time invariant charge density, we can check that $\rho(t) d\mathbf{r}(t) = \rho d\mathbf{r}_0$ and $\rho(t_1) d\mathbf{r}(t_1) = \rho d\mathbf{r}_{10}$. These relationships can be used to simplify the double-fold integrations in evaluating the self-force and the self-energy.

Substituting (32) into (22) and making use of (9) and (10), we can calculate the self-force of the electron numerically. Some derivations and formulae can be found in the Appendix.

As an example, we have calculated the self-force of the electron that is uniformly accelerated from $v = 0$ to $v = c_0$. The results for the self-force under different accelerations are plotted in Fig. 2. It can be seen that the self-forces have similar behavior. For large accelerations, the self-forces increase almost monotonically with the velocity and approach infinitely large when $v \rightarrow c_0$. For small accelerations, jitters may occur in the curves because of numerical errors.

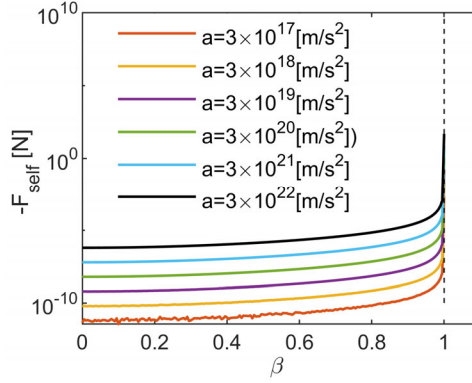


Fig. 2 The self-force of the electron under different accelerations.

For comparison, we have also calculated the self-force using the charge ball model for the electron at $a = 3 \times 10^{20} [\text{m/s}^2]$. The results are shown in Fig. 3. The overall behavior is very similar with a small discrepancy.

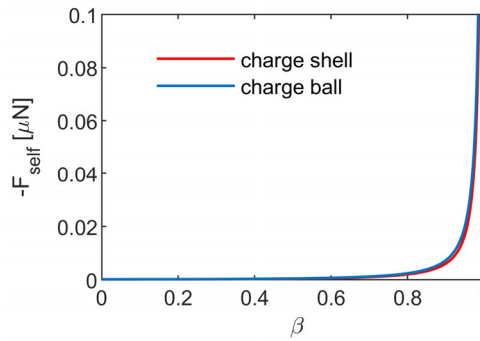


Fig. 3 The self-forces of the electron using two models. $a = 3 \times 10^{20} [\text{m/s}^2]$.

Substituting (32) into (24) and (25), we can calculate the self-energy (Coulomb-velocity energy) of the electron numerically. (See Appendix).

The results for the self-energy of the electron under a uniform acceleration of $a = 3 \times 10^{20} [\text{m/s}^2]$ are plotted in Fig. 4. It can be seen that the result using the charge ball model and that using the charge shell model have quite similar behavior. Just like the self-force, the self-energies also approach infinitely large when the velocity approaches c_0 . We may intuitively consider that the Coulomb-velocity energy is compressible. It is flexibly stored in the vacuum through its energy density. When the electron moves

faster, more energy can be piled up in the space, in other word, the energy density increases because of the compression effect and the total self-energy increases. In particular, the self-energy of the electron approaches infinitely large when $\beta \rightarrow 1$. Naturally, it requires an infinitely large force to further accelerate the electron.

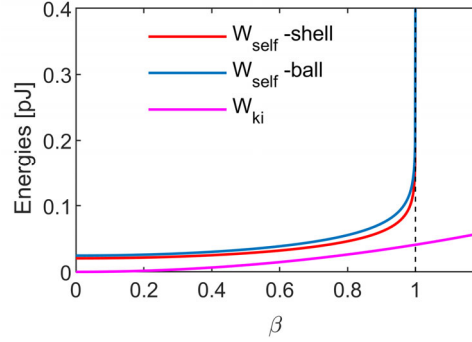


Fig. 4 The self-energy and the kinetic energy of the electron.

We have shown in [8] that there is an electromagnetic barrier at $\beta = 1$ for charged particles. When an electromagnetic source approaches the electromagnetic barrier, it will emit strong electromagnetic radiations. Electromagnetic shock waves occur when the source cross the electromagnetic barrier, just like sonic shock wave occurs when a jet plane crosses the sonic barrier in the air [6][8][12]. The wave front lags behind the source. The field is confined in the conical zone and cannot surpass the source. It has also been shown that at the edges of the shock wave zone, the amplitudes of the fields tend to become infinitely large. The electric field distributions for the uniformly moving charge is shown in Fig.5.

In this paper, we have numerically demonstrated that the self-force and the self-energy also approach infinitely large when the source approaches the electromagnetic barrier. For comparison, we have also plotted the kinetic energy $W_{ki} = 0.5m_e v^2$ of the electron in Fig. 4. Obviously, it does not approach infinite large unless $\beta \rightarrow \infty$.

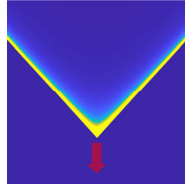


Fig.5. Cherenkov radiation of a moving charge at $\beta = 1.5$.

As a short summary, we remark that:

- 1) The electromagnetic barrier at $v = c_0$ is valid for charged objects but not for neutral objects with no charges. If there exist no other limitations, neutral objects can cross the electromagnetic barrier and move faster than the light in the vacuum. It may be not correct to take the light velocity in the vacuum as the upper limit of velocity of all objects in the universe.
- 2) We have not plotted the self-force and the self-energy of the electron for $\beta > 1$ in the figures because we have checked that although the self-energy is bounded but the self-force tends to be infinitely large if we adopt the charge ball or shell model for the electron. It might be natural to speculate that all charged objects cannot move faster than the light in the vacuum, so it is not necessary to explore the behavior for $\beta > 1$. However, since a jet plane can make supersonic flight safely, it is also not ridiculous to speculate that superluminal electromagnetic sources can exist in the universe. For example, they can cross the electromagnetic barrier successfully under help of other kind of forces, or just simply can be generated from other superluminal neutral objects.
- 3) The self-energy can be approximately treated as a basic property of the electron. However, different from the charge e and the rest mass m_e , the self-energy of the electron spreads over the space. The self-energy may be considered as compressible because the energy density changes with velocity. When the velocity increases, the field intensity increases, hence, the energy density increases. That is, more electromagnetic energy can be piled into the space to get a higher self-energy when the electron moves faster.
- 4) The uniform charge shell/ball model may be not proper anymore for the electron for $\beta > 1$. The development of the electron model for $\beta > 1$ needs further investigating. A possible candidate may be the circular disk model.

III. INTERPRETATION OF THE BERTOZZI EXPERIMENT

An experimental determination of the speeds of electrons accelerated by electric fields was carried out by Bertozzi in 1964 [13][14]. The principle of the experiment is outlined in Fig. 6. Electron bursts were emitted from a van de Graaff generator with kinetic energies up to 1.5 MeV. Each burst lasts about 3ns. The burst can be further accelerated by an electron linear accelerator (Linac) [15]. Two short tubes are placed near the Linac. The distance between the two tubes is about 8.4m, which is also the flight distance of the electrons. A voltage pulse will be induced when the electron burst pass the tube. The two signals transfer to the display through two cables with exactly the same time delay. The times of flight were measured by the displaying signals from the first and the second detectors on an oscilloscope screen. The electrons are finally absorbed by an aluminum disc and their energies were measured using a direct thermal technique with experimental error of 10%. The normalized kinetic energy $E_k/m_e c_0^2$ and the measured velocity were recorded. For the purpose of comparison, we cite the data of 4 measurement points from [13] and listed them in Table 1.

Table 1: Parameters of the electron

E_k (Mev)	$E_k/m_e c_0^2$	β^2
0.5	1	0.752
1.0	2	0.828
1.5	3	0.922
4.5	9	0.974

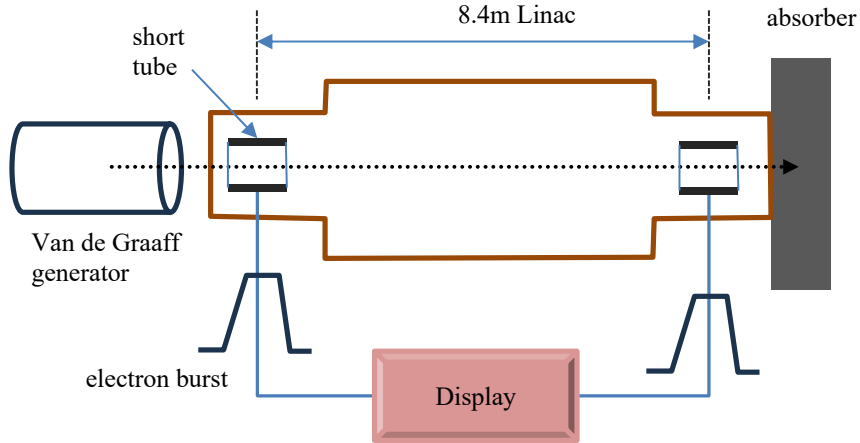


Fig. 6 Schematic diagram of Bertozzi experiment set up [13].

In Bertozzi experiment, the kinetic energy was measured using a thermal technique that can test the energy of the electrons absorbed by the aluminum absorber. It is reasonable to consider that the tested energy includes the electromagnetic self-energy (Coulomb-velocity energy) $W_{\rho J}$ and the kinetic energy W_{ki} . It does not include the emitted energy W_{emit} because it has already left the electron before being absorbed. We have numerically calculated the total energy, i.e., $W_{tot} = W_{\rho J} + W_{ki}$ with parameters of the electron in Table 2. As can be seen in (25), the numerical results of the energies are dependent on the radius r_e of the electron. We have adopted two values of r_e for the sake of comparison. One is the classical electron radius obtained by equating its classical electrostatic energy to its rest energy, which is $r_e = 2.817 \times 10^{-15}$ m. The second value is $r_e = 1.0 \times 10^{-15}$ m, which is chosen between the classical radius and the radius (9.1×10^{-17} m) obtained by scaling the radius of the proton. We can see that the results may be different by using different values of r_e . We also added in Fig. 7 the results obtained based on the theory of special relativity (SR), as proposed in [13]. It can be seen that although the measured data agree with the SR better than the numerical results that we have obtained, however, the most important point we want to highlight is that the velocity limit for the electrons when they are accelerated using a Linac can be indeed predicted according to the classical Maxwell's theory. We cannot conclude that the experiment results have provided a solid support to the effect of special relativity that the light velocity in the vacuum is the upper limit of the velocity of all objects in the universe in all frames, as was claimed in many text books.

Table 2: Parameters of the electron

Charge e [C]	1.602176634e-19
Mass m_e [kg]	9.10e-31
Radius r_e [m]	2.817e-15
	1.0e-15

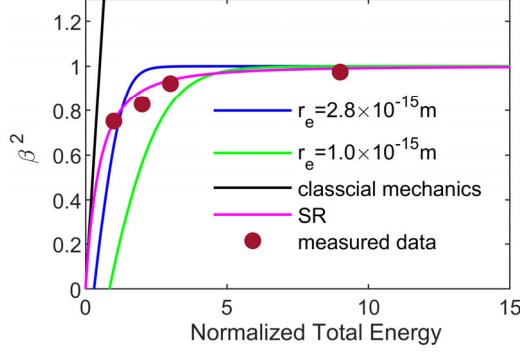


Fig. 7 Comparison of the energies. r_e : numerical results with different electron radius. SR: with special relativity. The measurement data are cited from [13].

IV. CONCLUSIONS

This is the third paper of my four-preprint series for revisiting the classical electromagnetic theory. The explicit expressions for the self-force and the self-energy of a moving electron can provide accurate and wide-span descriptions for the behavior of the electron, especially when its velocity approaches that of the light in the vacuum. The numerical results clearly show that the self-force and the self-energy approach infinitely large when $v \rightarrow c_0$. The conventional formulations, like the Larmor formula, the Abraham-Lorentz motion equation, failed to catch this scenario because they are basically valid only for low velocity situations. Moreover, conventional formulations tend to handle the electromagnetic problems by analogous to the Newtonian mechanics, and try to describe the behavior of electromagnetic sources in a way similar to that of the mechanical system. The essential fact that electromagnetic systems obey the Maxwell's theory instead of the Newtonian mechanics might have not been taken seriously enough. The Bertozzi experiment and other experiments may have been misinterpreted [16], so is the electromagnetic barrier at $v = c_0$. Consequently, the light velocity in the vacuum is incorrectly set as the upper velocity limit for all objects in the universe. However, numerical results from the classical Maxwell's theory show that c_0 is at least not the upper velocity limit for neutral objects, and may be broken even for charged particles if they can cross the electromagnetic barrier with external help.

APPENDIX

Assume that for $\beta < 1$ the electron is a charge shell with radius r_e . Its center moves along a rectilinear trajectory $\mathbf{s}(t)$ in the vacuum. Denote $\mathbf{v}(t) = c_0 \boldsymbol{\beta}(t) = \dot{\mathbf{s}}(t)$, $\mathbf{a}(t) = \dot{\boldsymbol{\beta}}(t)$. The coordinate for the observation point and the source point can be expressed by

$$\begin{cases} \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{s}(t) \\ \mathbf{r}_1(t_1) = \mathbf{r}_{10} + \mathbf{s}(t_1) \end{cases} \quad (33)$$

For uniform acceleration, the distance between the source and field point can be expressed by

$$\mathbf{R} = \mathbf{r}_0 + \mathbf{s}(t) - \mathbf{r}_{10} - \mathbf{s}(t_1) = (\mathbf{r}_0 - \mathbf{r}_{10}) + \mathbf{s}(t) - \mathbf{s}(t_1) = \mathbf{R}_0 + \mathbf{v}(t)(t - t_1) - \frac{1}{2} \mathbf{a}(t)(t - t_1)^2 \quad (34)$$

Because $R = c_0(t - t_1)$, we can rewrite (34) as

$$\mathbf{R} = \mathbf{R}_0 + \boldsymbol{\beta}(t) c_0(t - t_1) - \frac{1}{2c_0} \dot{\boldsymbol{\beta}}(t) c_0^2(t - t_1)^2 = \mathbf{R}_0 + R \boldsymbol{\beta}(t) - \frac{1}{2c_0} \dot{\boldsymbol{\beta}}(t) R^2 \quad (35)$$

From which the equation for R is derived to be

$$-\frac{1}{4c_0^2} \dot{\boldsymbol{\beta}}(t) \cdot \dot{\boldsymbol{\beta}}(t) R^4 + \frac{1}{c_0} \boldsymbol{\beta}(t) \cdot \dot{\boldsymbol{\beta}}(t) R^3 + \left[1 - \boldsymbol{\beta}(t) \cdot \boldsymbol{\beta}(t) + \frac{1}{c_0} \mathbf{R}_0 \cdot \dot{\boldsymbol{\beta}}(t) \right] R^2 - 2\mathbf{R}_0 \cdot \boldsymbol{\beta}(t) R - R_0^2 = 0 \quad (36)$$

For a given trajectory, $\boldsymbol{\beta}(t)$ and $\dot{\boldsymbol{\beta}}(t)$ are known. Solving (35) at time t , we get R , and then t_1 if necessary. The potentials and fields at the point (\mathbf{r}, t) in the sphere V_e can be evaluated, hence, the self-energy and self-force can be calculated. Note that for uniform motion, $\dot{\boldsymbol{\beta}}(t) = 0$, (36) is reduced to a second order linear equation and can be solved analytically. Otherwise, we have to resort to efficient numerical method to solve a fourth-order linear equation.

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